DEFINITE INTEGRATION

ELEMENTARY EXERCISE

Sol.1 $\int_0^1 \frac{dx}{3\sqrt{\left(\frac{4}{3}\right)^2 + x^2}} + \int_0^2 \frac{dx}{2\sqrt{\left(\frac{3}{2}\right)^2 + x^2}}$

$\frac{1}{3}\left(\ln/ x + \sqrt{\frac{16}{9} + x^2}\right)_0^1 + \frac{1}{2}\left(\ln| x\sqrt{\frac{9}{4} + x^2}\right)_0^2$

on putting limits.

$\frac{1}{3}$ ln 2 + 1/2 ln 3 = lna

ln $2^{1/3}$ + ln $3^{1/2}$ = lna

a = $2^{1/3}$. $3^{1/2}$

Sol.2 put –x = t

–dx = dt

$\int_0^{-\ln 2} te^t dt$

IBP.

$\left[tet - e^t\right]_0^{-\ln 2}$

on putting limits

$\frac{1}{2}$ (1 – ln2) = $\frac{1}{2}$ ln (e/2)

Sol.3 Taking L.C.M.

$\int_1^e \frac{1 + \ln x}{\sqrt{x \ln x}} dx$

put XLNx = t^2

(1 + lnx)dx = 2t dt

$\int_0^{\sqrt{e}} \frac{2tdt}{t} = 2[t]_0^{\sqrt{e}} = 2\sqrt{e}$

Sol.4 $\int_{\pi/2}^{3\pi/2} f(x)dx$ IBP.

$\left[f(x).x\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} f'(x).x.dx$

$= 2 - \frac{\pi}{2}(a - 3b)$

Sol.5 put 5 – 4x = t^2

–4dx = 2t.dx

dx = $\frac{-t}{2}$ dt

$\int_1^3 \frac{\left(\frac{5 - t^2}{4}\right)(t/2)}{t} dt$

$\frac{1}{8}\int_1^3 5 - t^2 dt = 1/6$

Sol.6 put lnx = t

x = e^t

dx = e^t dt

$\int_{\ln 2}^1 e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt$

standard integeral

$\left(e^t \frac{1}{t}\right)_{\ln 2}^1 = e - \frac{2}{\ln 2}$

Sol.7 divide num,and deno by $\cos^4 x$

$\int_0^{\pi/4} 2\frac{\tan x \sec^2 x}{1 + \tan^4 x}$

put tanx = t

$\sec^2 x$ = dt

$\int_0^1 \frac{2t}{1 + t^4} dt$

put t^2 = u

$\int_0^1 \frac{du}{1 + u^2} = \left(\tan^{-1}\right)_0^1 = \pi/4$

Sol.8 $\int_0^1 \frac{dt}{(1 + t)}(2 + t) = \frac{A}{1 + t} + \frac{B}{2 + t}$

A = 1 B = 1

$\int_0^1 \frac{1}{1 + t} - \int_0^1 \frac{1}{2 + t} = \ln 4/3$

Sol.9 divide num,and deno by $\cos^6 x$

$$\int_0^{\pi/4} \frac{\tan^2 x.\sec^2 x.dx}{(1+\tan^3 x)^2}$$

put $1 + \tan^3 x = t$

$$1/3\int_1^2 \left(\frac{dt}{t^2}\right) = 1/6$$

Sol.10 Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int_{\tan^{-1} 1/3}^{\tan^{-1} 3} \theta\, \text{cosec}^2\theta\, d\theta$$

now solve by parts

Sol.11 $\int_2^3 \frac{dn}{\sqrt{-x^2+6x-5}}$

$$\int_2^3 \frac{dx}{\sqrt{(2)^2-(x-3)^2}} = \pi/6$$

Sol.12 put $x = \cos^2\theta + 3\sin^2\theta$

$dx = 2\sin 2\theta\, d\theta$

$$2\int_{\pi/6}^{\pi/4} \tan\theta.\sin 2\theta.d\theta$$

$$4\int_{\pi/6}^{\pi/4} \sin^2\theta d\theta = \frac{\sqrt{3}}{2} - 1 + \frac{\pi}{6}$$

Sol.13 $\frac{1}{2}\int_0^{\pi/4} x\cos 4x + 1/2\int_0^{\pi/4} x\cos 2x dx$

= apply ibp

$= 1/16\,(\pi - 3)$

Sol.14 put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

$$= \int_0^{\pi/4} \frac{\sec^2 x/2dx}{5+5\tan^2 x/2 + 8\tan x/2}$$

put $\tan x/2 = t$

$$\frac{2}{5}\int_0^1 \frac{1}{(t+8/10)^2+(6/10)^2}$$

$$= 2/3\tan^{-1}\left(\frac{10t+8}{6}\right)_0^1 = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

Sol.15 Put $x = \frac{1}{t}$ $\qquad dx = \frac{-1}{t^2}\, dt$

$$\int \frac{-1/t^2.dt}{\left(1-\frac{2}{t^2}\right)\sqrt{\frac{t^2-1}{t^2}}} \Rightarrow \int \frac{t.dt}{(t^2-2)\sqrt{t^2-1}}$$

put $t^2 = u$ $\qquad 2t\,.dt = du$

$$-\int \frac{dt}{(u-2)\sqrt{u-1}}$$

now put $u - 1 = v^2$

$$-\int \frac{1}{v^2-1}.dv$$

$$= \frac{-1}{2}\ln\left|\frac{v-1}{v+1}\right|$$

Sol.16 $\int_0^{\pi/2} \frac{dx}{1+\cos\theta\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$

put $\tan x/2 = t$

and get $\sec^2 x/2\, dx = 2dt$

solve and get $= \frac{\theta}{\sin\theta}$

Sol.17 $\int \frac{e^x}{e^{2x}+1} + \int \frac{1}{e^{2x}+1} dx$

$$\int \frac{e^x}{e^{2x}+1} \quad + \quad \int \frac{e^{-2x}}{e^{-2x}+1} dx$$

put $e^x = t$ $\qquad$ put $e^{-2x} + 1 = t$

$e^x\, dx = dt$ $\qquad$ $-2e^{-2x}\, dx = dt$

$= 1/2\,(\pi/6 + \ln 3 - \ln 2)$

Sol.18 put $1 - \sin 2x = t$

$-2\cos 2x\, dx = dt$

$$= +\frac{1}{2}\int_0^1 \sqrt{t}dt = 1/3$$

Sol.19 put $x = 3\sin^2\theta$

$dx = 6\sin\theta\cos\theta\, d\theta$

$$3\int_0^{\pi/2} (\sin^2\theta) d\theta = 3\pi/2$$

Sol.20 put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\int_0^{\pi/6} \sec^2\theta\, d\theta$$

$$= \frac{1}{2} \ln |2 + \sqrt{3}|$$

Sol.21 take x^4 common

$$\int_1^2 \frac{dx}{x^5(1+x^{-4})}$$

put $1 + x^{-4} = t$

and get $-4x^{-5}\,dx = dt$

$$= \frac{1}{4} \ln \frac{32}{17}$$

Sol.22 put $\sin\phi = t$

$$\int_0^1 t\sqrt{a^2t^2 + b^2 - b^2t^2}\,.dt$$

now put $t^2 = u$

$$\frac{1}{2}\int_0^1 \sqrt{(a^2 - b^2)u + b^2}\,.du$$

Sol.23. (a) $\displaystyle\int_0^{3\pi/4} (\sin x + x\sin x + \cos x - x\cos x)dx$

$$\int_0^{3\pi/4} (\sin x + \cos x)dx + \int_0^{3\pi/4} x(\sin x - \cos x)dx$$

apply ibp

$$= 2(\sqrt{2} + 1)$$

Sol.23 (b) divide & mul. by x

$$\int_{\pi/2}^{\pi} x^{\sin x + 1}\left(\frac{1}{x} + \cos x.\ln x + \frac{\sin x}{x}\right)dx$$

put $x^{\sin x+1} = t$

and get $x^{\sin x+1}\left(\frac{1}{x} + \cos x.\ln x + \frac{\sin x}{x}\right)dx = dt$

$$= \pi - \pi^2/4$$

Sol.24 put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

further solve by ibp and get

$$\frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2}\ell n2$$

Sol.25 $\displaystyle\int_0^1 e^{-2x} f(x).dx = 3$

solve by parts

$$\left(f(x)\frac{e^{-2x}}{-2}\right)_0^{\ln 2} + \frac{1}{2}\int_0^{\ln 2} f'(x)e^{-2x}dx = 3$$

now put the limits

$= 13$

Sol.26 $\displaystyle\int \frac{dx}{(x + \cos\alpha) + (1 - \cos\alpha)}$

$$\Rightarrow \int \frac{dx}{(x + \cos\alpha) + \sin\alpha}$$

$$\Rightarrow \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x + \cos\alpha}{\sin\alpha}\right)\right]$$

$$\Rightarrow \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{1 + \cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$\Rightarrow \frac{1}{\sin\alpha}\tan^{-1}\left(\tan\frac{\alpha}{2}\right) \Rightarrow \frac{\alpha}{2\sin\alpha/2}$$

Sol.27 $\ln |x + \sqrt{1 + x^2}\,|_a^b$

$\ln |b + \sqrt{1 + b^2}| - \ln |a + \sqrt{1 + a^2}|$

on puting values get the answer

Sol.28 put $x^x = t$

& get $x^x(1 + \ln x)dx = dt$

$= 0$

Sol.29 put $x^2 = \cos^2\theta$

$2x\,dx = -2\sin 2\theta\, d\theta$

$$\int_0^{\pi/4} 2\cos^2 2\theta.\cos^2\theta\ d\theta$$

$$\Rightarrow \int_0^{\pi/4} \cos^2 2\theta + \cos^3 2\theta\ d\theta$$

$$\Rightarrow \int_0^{\pi/4}\left[\frac{1+\cos 4\theta}{2}\right]d\theta + \int_0^{\pi/4}\left[\frac{\cos 6\theta + 3\cos 2\theta}{4}\right]d\theta$$

$$\Rightarrow \frac{3\pi+8}{24}$$

Sol.30 $\int_0^1 f(x)g'(x) + f'(x)g(x) - \int \frac{f(x)g'(x) - f'(x)g(x)}{g^2(x)}$

$$\int \frac{d}{dx}\left(f(x)g(x)\right) + \int \frac{d}{dx}\frac{f(x)}{g(x)}$$

= 2009

Sol.31 $\int \frac{\sin\theta+\cos\theta}{\frac{9}{16}+1-1+\sin 2\theta} = \int \frac{\sin\theta+\cos\theta}{\left(\frac{5}{4}\right)^2 - (\cos\theta - \sin\theta)^2}$

= now put $\cos\theta - \sin\theta = t$

and get $(\sin\theta + \cos\theta)\,d\theta = -dt$

$$\int_0^1 \frac{dt}{\left(\frac{5}{4}\right)^2 - (t)^2} = \frac{1}{20} \ln 3$$

Sol.32 $\int_0^{\pi} \theta\cos\theta\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$$\Rightarrow \frac{1}{2}\left\{\int_0^{\pi}\theta.\cos\theta - \frac{1}{2}\int_0^{\pi}\theta\left(\cos\theta + \cos 3\theta\right)d\theta\right\}$$

$$\Rightarrow \frac{1}{4}\left\{\int_0^{\pi}\theta.\cos\theta - \int_0^{\pi}\theta\cos 3\theta\,d\theta\right\}$$

Solve by part to get solution

Sol.33 divide num. & Deno by $\sin^{2x}$

$$\int_0^{\pi/2}\frac{\operatorname{cosec}^2 x + 2\operatorname{cosec}x\cot x}{(2\operatorname{cosec}x + \cot x)^2}$$

now put $(2\operatorname{cosec}x + \cot x) = t$

$(\operatorname{cosec}^2 x + 2\operatorname{cosec}x\cot x)\,dx = -dt$

$= \frac{1}{2}$

Sol.34 $\int_0^{\pi/2}\frac{x}{1+\cos x} + \int_0^{\pi/2}\frac{\sin x}{1+\cos x}$

$$1/2\int_0^{\pi/2} x\sec^2 x/2 + \int_0^{\pi/2}\frac{\sin x}{1+\cos x}$$

apply ibp

$= \frac{\pi}{2}$

Sol.35 solve by partial fraction

$$\int_{3/4}^{4/3}\frac{2x^2+x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

solve & get A = 1, B = 1, C = 0

$$\int_{3/4}^{4/3}\frac{1}{x+1} + \int_{3/4}^{4/3}\frac{x}{x^2+1}$$

on solving get = 2ln |4/3|

Sol.36 $\int_0^1 \frac{1}{x+1(\sqrt{1-x^2})} + \int_0^1 \frac{(1-x^2)}{(1+x)\sqrt{1-x^2}}$

put x + 1 = 1/t put x = cosθ

$= \frac{\pi}{2}$

Sol.37 Function is not defined at x = 0.

$$\int_{-1}^{0^-}\left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx + \int_{0^+}^{1}\left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx$$

$$\Rightarrow \left(\frac{1}{1+e^{1/x}}\right)_{-1}^{0^-} + \left(\frac{1}{1+e^{1/x}}\right)_{0^+}^{1}$$

= 2/1+e

Sol.38 $\int \frac{dx}{x(\ln x + 1)}$

put ln x + 1 = t

1/x dx = dt

$$\int_1^2 \frac{dt}{t} = \ln 2$$

Sol.39 Let $\left(\frac{3\pi}{8}-\frac{x}{4}\right) = A$

let $\left(\frac{11\pi}{8}+\frac{x}{4}\right) = B$

$$\int_0^{\pi} \cos^2 A - \cos^2 B = \int_0^{\pi} \sin(A+B).\sin(A-B)$$

$$\int_0^{\pi} \sin\left(\frac{14\pi}{8}\right).\sin(x/2) = \sqrt{2}$$

Sol.40 $\int_0^{\pi} f(x)\sin x + \int_0^{\pi} f''(x)\sin x = 5$

apply IBP

$$(-f(x)\cos x)_0^{\pi} + \int_0^{\pi} f'(x)\cos x + \int_0^{\pi} f''(x)\sin x = 5$$

f(0) = 3

Sol.41 $\int_a^b \mathrm{sgn}(x).dx$

$(\mathrm{sgn}(x).x)_a^b$ = b. sgn (b) – a sgn (a)

= |b| – |a|

Sol.42 it is AGP

$$f(x) = e^{-x} + 2e^{-2x} + 3e^{-3x} + ...\infty$$

$e^{-x} f(x) = \quad e^{-2x} + 2e^{-3x} + ...\infty$

$f(x)\ (1-e^{-x}) = e^{-x} + e^{-2x} + e^{-3x} + ...\infty$

$= e^{-x}/1-e^{-x}$ (sum of ∞ G.P)

$$f(x) = \frac{e^{-x}}{(1-e^{-x})^2} = 1/2$$

Sol.43 multiply divide by (sec x – tan x)

$$\int_0^{\pi/2} (\sec x - \tan x)\ \frac{\mathrm{cosec}\ x}{\sqrt{1+2\,\mathrm{cosec}\ x}}\ dx$$

$$\int_0^{\pi/2} \frac{\sec x(\mathrm{cosec}x - 1)}{\sqrt{1+2\,\mathrm{cosec}\ x}} dx$$

put 1 + 2 cosec x = t^2

Sol.44 by IBP

$$x\int f''(x) - \int\left(\frac{d}{dx}x\right)\int f''(x).dx$$

$$(xf'(x))_0^1 - \int f'(x).dx$$

$$\left(\frac{-x\sin(\tan^{-1}x)}{1+x^2}\right)_0^1 - (\cos\tan^{-1}x)_0^1$$

$$= 1 - \frac{3}{2\sqrt{2}}$$

Sol.45 (a) If f & g are inverse then

$$\int_a^b f(x)dx + \int_c^d g(y)dy = bd - ac$$

then,

$$\int_1^5 f(x)dx + \int_2^{10} g(y)dy = 5 \times 10 - 1 \times 2$$

= 48

Sol.45 (b) $\int_0^1 f(x)dx + \int_0^1 f^{-1}(y)dy = 1$

$$1/3 + \int_0^1 f^{-1}(y)dy = 1$$

= 2/3